Sample Problems AB #5 AB #5

II.B ① Separating Differential Equations and the equation $y = Ce^{kt}$

1. The rate of change of the number of coyotes $N(t)$ in a population is directly porportional to $650 - N(t)$, where t is the time in years. When $t=0$, the population is 300, and when $t=2$, the population has increased to 500. Find the population when $t=3$.

(this problem can be found on page 425 of our txtbk)

2. The rate of change of V is proportional to V. When $t=0$, $V=20{,}000$ and when $t=4$, $V=12{,}500$. What is the value of V when $t=6$?

(this problem can be found on page 418, #23 of our txtbk)

3. Carbon-14 has a half-life of 5730 years. The amount left after 10,000 years is 2g. How much Carbon-14 was there initially?

(this problem can be found in our 6.2 notes).

4. A certain type of bacteria increases continuously at a rate proportional to the number present. If there are 500 present at a given time and 1000 present 3 hours later, how many hours (from the initial given time) will it take for the number of bacteria to be 2500?

(this problem can be found on our Differential Equations Quiz #3).

② The logistic equation and ~~the~~ ~~eqn~~ $P = e^{rt}$

1. A conservation organization releases 25 ~~Florida~~ panthers into a game preserves. After 2 years, there are 39 panthers in the preserve. The preserve has a carrying capacity of 200 panthers.

a) Find the population after 5 years.
b) When will the population reach 100?
c) When ~~will~~ ~~the~~ is the population growing most rapidly?

(this question can be found on page 401 #79 of our txtbk)

2. At $t=0$, a bacteria culture weighs 1g. Two hours later, the culture weighs 2g. The maximum weight of the culture is 10g. When will the culture's weight be 8g?
(this problem can be found on page 431 #80 of our txt

3. Find the annual rate of interest if \$750 is doubled in $7\frac{3}{4}$ years and is compounded continuously.
(This problem can be found on page 419, #45).

4. Find the time necessary for \$1000 to double if invested at a rate of 8.5% and is compounded continuously.

I The Distance Travelled by a particle along a line

1. A particle moves along the x-axis so that at time t its position is given by
$$x(t) = t^3 - 6t^2 + 9t + 11$$

a) At $t=0$, is the particle moving to the right or the left? Why?
b) At $t=1$, is the velocity of the particle increasing or decreasing? Why?
c) ~~Find the total distance traveled by the particle over the time interval $0 \le t \le 5$.~~

2. The rate of change, of the altitude of a helicopter is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where t is measured in hours. Assume the balloon is initially at ground level.

a) For what values of t, $0 \le t \le 4$, is the altitude of the ~~helicopter~~ helicopter decreasing?
b) What is the altitude of the ~~balloon~~ helicopter when it is closest to the ground during the time interval $2 \le t \le 4$?
c) Find the value of $\int_0^4 r(t)\,dt$ and explain the meaning of the answer in the context of the question.
d) Find the value of $\int_0^4 |r(t)|\,dt$ and explain the meaning of the answer in the context of the question.

Solution Key to sample problems part II B and I

① Separating Differential Equations and the equation $y = Ce^{kt}$

1. The rate of change is proportional to $650 - N(t)$, we can say:

$$\frac{dN}{dt} = k(650 - N)$$

We then solve by putting the variables one side and then integrating

$$\int \frac{dN}{(650-N)} = \int k\,dt \implies -\ln|650 - N| = kt + C$$

Then we solve for N:

$$650 - N = e^{-kt-C} \implies N = 650 - Ce^{-kt}$$

That is the general solution.

Because we know that $N = 300$ when $t = 0$, we can solve for C:

$$650 - (300) = Ce^{0} \quad C = 350 \implies N = 650 - 350e^{-kt}$$

Then, to solve for k, we use our knowledge that when $N = 500$ when $t = 2$:

$$(500) = 650 - 350e^{-k(2)}$$

$$e^{-2k} = \frac{3}{7} \implies k \approx 0.4236$$

$$\implies N = 650 - 350e^{-0.4236t}$$

When $t = 3$ we can approximate the population to be:

$$N = 650 - 350e^{-0.4236(3)} \implies N \approx \boxed{552 \text{ coyotes}}$$

2. We begin by setting up the equation, putting the variables on one side and then integrating:

$$\frac{dV}{dt} = kV \Rightarrow \int \frac{dV}{V} = \int k\,dt \Rightarrow \ln|V| = kt + C$$

$$V = Ce^{kt}$$

We solve for C knowing that $V = 20000$ when $t = 0$:

$$2000 = Ce^{0} \qquad C = 20000 \Rightarrow V = 20000e^{kt}$$

We solve for k knowing that $V = 12500$ when $t = 4$:

$$(12500) = 20000\,e^{k(4)} \qquad k \approx -0.1175$$

$$V = 20000e^{-0.1175t}$$

To find V when $t = 6$, we plug and chug:

$$V = 20000e^{-0.1175(6)} \approx \boxed{9882}$$

3. We will use the equation $y = Ce^{kt}$ because we know that half of the initial Carbon-14 amount will be gone in 5730 years, we can say that:

$$\cancel{y(5730)} \quad y = \tfrac{1}{2}C \text{ when } t = 5730$$

Thus: $\frac{1}{2}\cancel{C} = \cancel{C}e^{5730k}$

$$\ln|\tfrac{1}{2}| = 5730k \qquad k \approx -0.00012097$$

To find the initial amount, we solve for C:

$$2 = Ce^{-0.00012097(5730)}$$

$$C \approx \boxed{6.705g}$$

4. We will use the equation $P = Ce^{kt}$.
We know that $P = 500$ when $t = 0$ so we can solve for C:

$$500 = Ce^{0} \qquad C = 500$$

We can then solve for k knowing that when $P = 1000$, $t = 3$:

$$1000 = 500e^{3k} \qquad k \approx 0.231$$

To solve for the ~~am~~ time when the amount is 2500, we plug and chug:

$$2500 = 500e^{0.231t}$$

$$t \approx \boxed{6.97 \text{ hours}}$$

2) The logistic equation and $P = e^{rt}$

1 a) We will use the logistic equation,

$$y = \frac{L}{1 + be^{kt}}$$

We know that $L = 200$ and $y = 25$ when $t = 0$, so we can solve for b:

$$25 = \frac{200}{1 + be^{0}} \qquad b = \frac{200}{25} - 1 = 7$$

We use $y = 39$ when $t = 2$ to solve for k:

$$39 = \frac{200}{1 + 7e^{2k}} \qquad k \approx -0.264$$

Then we plug in $t = 5$ to find the population after 5 years:

$$y = \frac{200}{1 + 7e^{0.264(5)}} \approx 7.35 \approx \boxed{7 \text{ panthers}}$$

b) We plug in $y = 100$ and solve for t:

$$100 = \frac{200}{1 + 7e^{0.264t}} \qquad t \approx \boxed{7.37 \text{ years}}$$

c) The population will be increasing the fastest when $t = \frac{1}{2}L$

so: $\frac{1}{2}L = \frac{\text{[illegible]}}{100}$

$\boxed{\text{The population will be increasing the fastest in 100 years.}}$

2. We first set up the logistic equation:

$$y = \frac{10}{1+be^{-kt}}$$

We solve for b:

$$1 = \frac{10}{1+be^{0}} \quad b = 9$$

We solve for k:

$$2 = \frac{10}{1+9e^{-2k}} \quad k \approx 0.405$$

Then we find the value of t when $y = 8$

$$8 = \frac{10}{1+9e^{-0.405t}}$$

$$\boxed{t \approx 8.8 \text{ hours}}$$

3. We'll use the $P = e^{rt}$ equation:

$P = e^{rt}$ we know that $t = 7\frac{3}{4}$ when $P = 750 \times 2 = \$1500$

$1500 = e^{r(7\frac{3}{4})}$ we can solve for r to find the rate:

$r \approx 0.9436 \approx \boxed{0.944\%}$

4. Again, use $P = e^{rt}$ solve for t:

$P = \$1000 \times 2 = \$2000$ $\quad 2000 = e^{8.5t}$

$r = 8.5\%$

$t \approx \boxed{8.15 \text{ years}}$

I. The Distance Travelled by a Particle along a Line

1. a) when $t=0$, the particle is moving to the right because the velocity ($v(t) = x'(t)$) is positive:

$v(t) = x'(t) = 3t^2 - 12t + 9$

$x'(0) = 3(0)^2 - 12(0) + 9 = 9.$

b) When $t=1$ the velocity of the particle is decreasing, because the acceleration ($a(t) = v'(t) = x''(t)$) is negative:

$a(t) = 6t - 12 \quad a(0) = -12$

2. a) The altitude of the helicopter is decreasing whenever $r(t) \leq 0$. By graphing $r(t)$ and finding where it crosses the x-axis, we find that the time interval will be:

$$\boxed{1.571993 \leq t \leq 3.5141369 \text{ hours.}}$$

b) From our result in part (b), we know that the helicopter is closest to the ground at $t = 3.514$ hours.
If $y(t)$ denotes the altitude of the helicopter after t hours, then, by the Fundamental Theorem of Calculus, the minimum altitude of the helicopter for $2 \leq t \leq 4$ is

$$y(3.514) = y(0) + \int_0^{3.514} r(t)\,dt = \boxed{1.348 \text{ km}}$$

c) $\int_0^4 (r(t))dt = \boxed{2.667 \text{ km.}}$

This integral represents the net change in the altitude of the helicopter over the time ~~integral~~ interval $0 \leq t \leq 4$.
Since the helicopter began at ground level, this is also the altitude of the helicopter at $t = 4$ hours.

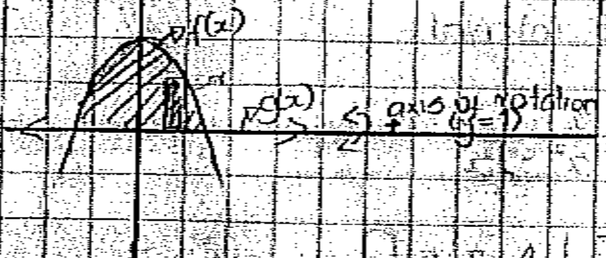
d) $\int_0^4 |r(t)|dt = \boxed{11.529 \text{ km}}$

This integral represents the total vertical distance travelled by the helicopter over the time interval $0 \leq t \leq 4$. (the helicopter may have risen and decended repeatedly during the 4 hours).

Disk Method

Examples

Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.



1. Graph
2. Draw the representative rectangle
3. Because axis of rotation is // to the x-axis, we integrate in terms of x. If the axis of rotation was // to the to the y-axis, we would have integrated wrt to y.
4. The $r(x)$ is the distance from the axis of rotation to the function.
5. $A = \pi \int_a^b r(x)^2 \, dx$

$= \pi \int [(2 - x^2) - (1)]^2 dx$

$= \pi \int (1 - 2x^2 + x^4) dx$

$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1}$

$= \frac{16\pi}{15}$

* If you are not given the interval of the graph, set the given equations (functions) equal to each other and find the intersecting points.

$2 - x^2 = 1$

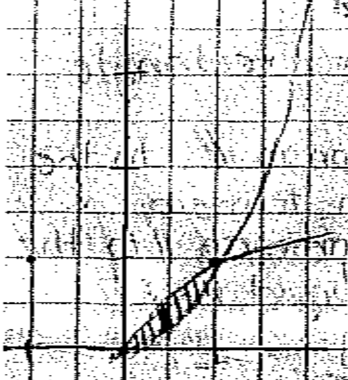
$1 - x^2 =$

$x = -1$, $x = 1$ } interval

Washer Method

Example

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.



1. Graph
2. Draw the rep. rectangle.
3. The axis of rotation is the x axis, therefore we integrate w.r.t to x.
4. Find the interval

$$\sqrt{x} = x^2$$
$$= x^2 - \sqrt{x}$$

$$x = 1 \quad x = 0$$

$R(x) \rightarrow$ outer radius

$r(x) \rightarrow$ inner radius

the distance between the axis of revolution

5. $V = \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2 \right] dx$

$$= \pi \int_0^1 \left[(x - x^4) \right] dx$$

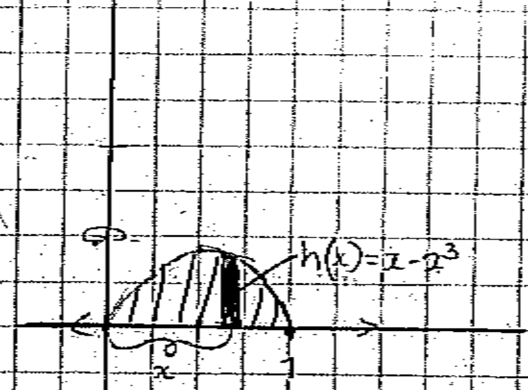
$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= \frac{3\pi}{10}$$

The Shell Method

Example

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \leq x \leq$ about the y-axis.



1. Graph
2. Because axis of rotation is vertical, use a vertical representative rectangle.
3. x is the variable of integration. The distance from the center of the rectangle to the axis of revolution is ~~$p(x) = x$~~.
4. $V = 2\pi \int_a^b r(x)\, h(x)\, dx$

$= 2\pi \int_0^1 x\,(x - x^3)\, dx$

$2\pi \int_0^1 x^2 - x^4\, dx$

$2\pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$

$= \frac{4\pi}{15}$

Sample Questions:

1. Find the area of the region between the graphs of $f(x) = 3x^3 - x^2$ and $g(x) = -x^2 + 2x$

2. Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$

Volume using Disk Method

1. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$ $x = 4$ about the line $y = 1$

2. Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$ $1 \leq y \leq 4$ about the y-axis.

Volume Using the Washer Method

1. Find the volume bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid.

2. Find the volume of a solid generated by revolving each region about the y-axis. The volume is bounded by $y = 2\sqrt{x}$, $y = 2$, $x = 0$

Find the volume of the region bounded by the parabola $y = x^2$ and the line $y = 2x$ ~~revolved~~ in the 1st quadrant is revolved about the y-axis.

Volume Using the Shell Method

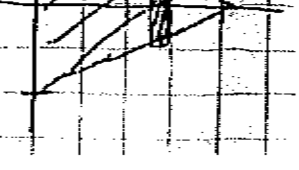
Find the volume of the region bound by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ revolved about the y-axis to generate a solid.

Find the volume of the region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$ revolved about the x-axis to generate a solid.

Volume of Solids with a known cross section

Find the volume of a solid. The base of the solid is the region bounded by the lines.

$f(x) = 1 - \frac{x}{2}$ $g(x) = -1 + \frac{x}{2}$ and $x = 0$



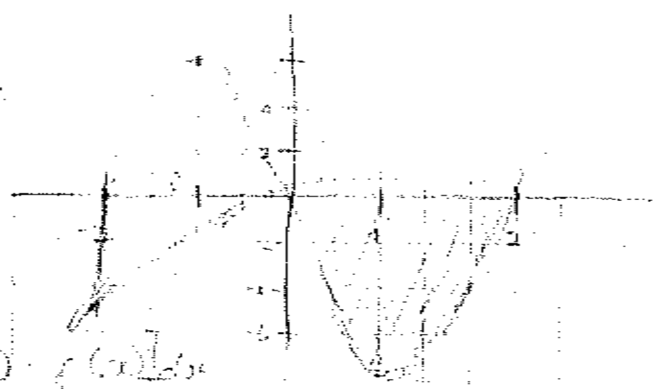
Sample Questions of Average Value of a Function:

1. Find the average value of $y = 2 + \sin(x)$ for $0 \leq x \leq 2\pi$

2. Find the average value of $y = -x + 2x - 5$ for $-3 \leq x \leq 2$.

$x^3 - 10x = -x^3 + 2x$
$3x^3 - 12x = 0$
$3x(x-2)(x+2) = 0$

$x = 0, 2, -2$



$$A = \int_{-2}^{0} [f(x) - g(x)]dx + \int_{0}^{2} [g(x) - f(x)]dx$$

$$= \int_{-2}^{0} (3x^3 - 12x)dx + \int_{0}^{2} (-3x^3 + 12x)dx$$ → plug into calc

$= 12 + 12$
$= 24$

$3 - y^2 = y + 1$
$0 = y^2 + y - 2$
$y = 1, -2$

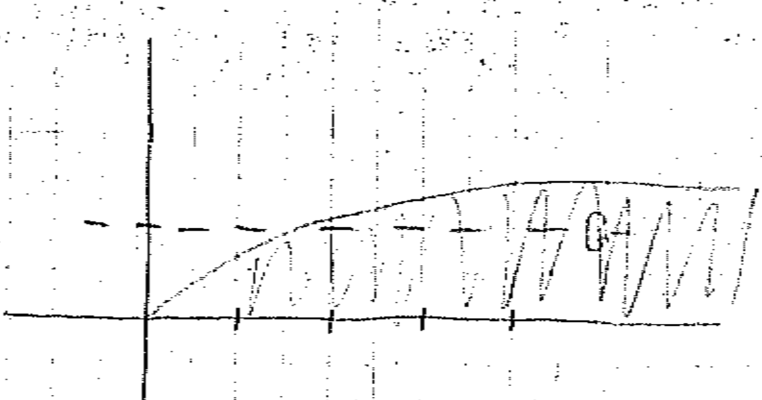
$$A = \int_{-2}^{1} (3 - y^2) - (y + 1)\, dy$$

$$= \int_{-2}^{1} (-y^2 - y + 2)\, dy$$

$$= \frac{9}{2}$$

Volume Using Disk Method

1. $$V = \int_{1}^{4} \pi [R(x)]^2 dx$$



$$= \int_{1}^{4} \pi (\sqrt{x} - 1)^2 dx$$

$$= \int_{1}^{4} \pi [x - 2\sqrt{x} + 1] dx$$

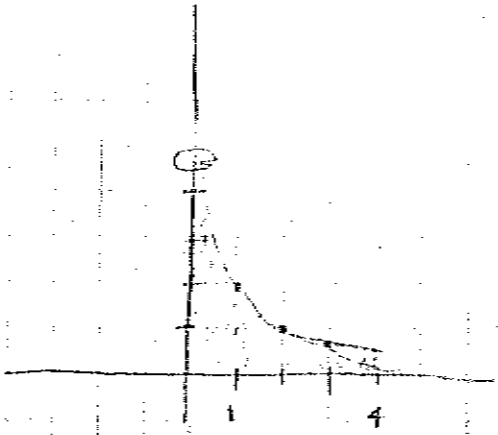
$$= \frac{7\pi}{6}$$

$$V = \int_1^4 \pi[R(y)]^2\,dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$= \pi\int_1^4 \frac{4}{y^2}\,dy$$

$$= 3\pi$$



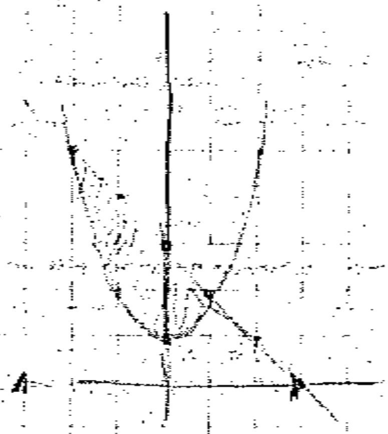
Using the Washer Method

$$V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx$$

$$= \int_{-2}^1 \pi\left((-x+3)^2 - (x^2+1)^2\right)dx$$

$$= \int_{-2}^1 \pi(8 - 6x - x^2 - x^4)\,dx$$

$$= \frac{117\pi}{5}$$

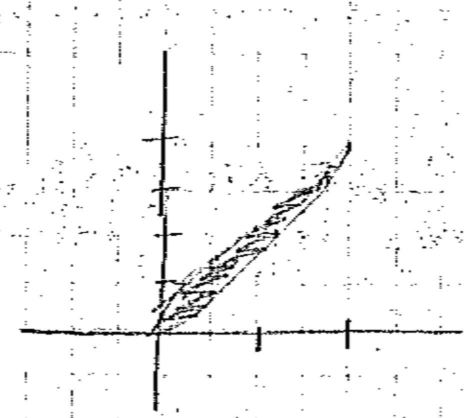


Find the volume of c

$$V = \int_0^4 \pi\left([\sqrt{y}]^2 - \left[\frac{y}{2}\right]^2\right)dy$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy$$

$$= \frac{8}{3}\pi$$

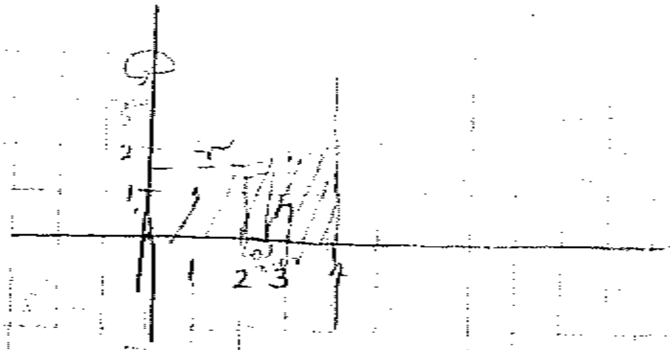


Volume Using Shell Method.

1. $V = \int_0^4 2\pi(\overset{r}{x})(\overset{h}{\sqrt{x}})\,dx$

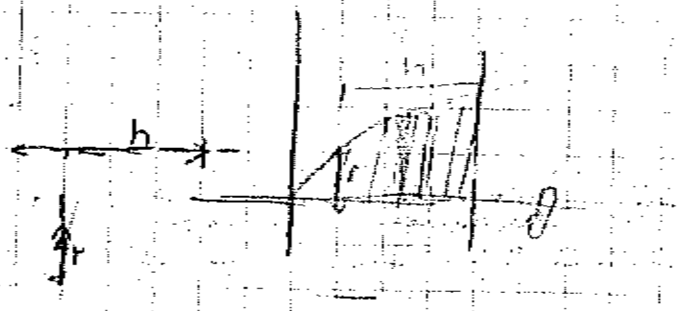
$= 2\pi \int_0^4 x^{3/2}\,dx$

$= \dfrac{128\pi}{5}$



$V = \int_0^2 2\pi(y)(4 - y^2)\,dy$

$= 8\pi$



Volume of known cross-sections.

Base $= \left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right) = 2 - x$

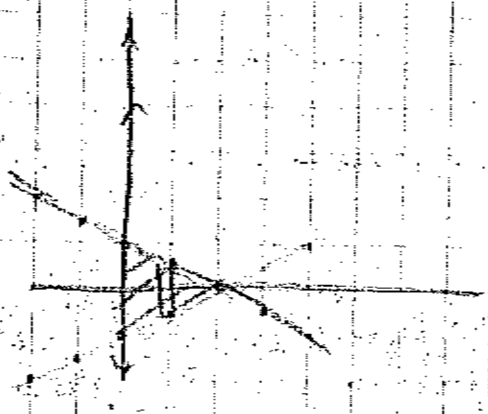
$A_{eq} = \frac{\sqrt{3}}{4} b^2$

$A(x) = \frac{\sqrt{3}}{4}(2 - x)^2$

$V = \int_0^2 \frac{\sqrt{3}}{4}(2 - x)^2\,dx$

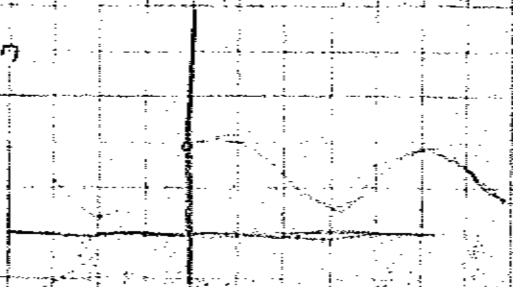
$= -\frac{\sqrt{3}}{4}\left[\frac{(2 - x)^3}{3}\right]_0^2$

$= \dfrac{2\sqrt{3}}{3}$



The average value of a function

1. $\frac{1}{2\pi - 0}\int_0^{2\pi} (2 + \sin(x))\,dx$



$= 2\ \pi$

2. Find average of $y = -x + 2x - 5$ for $-3 \le x \le 2$.

$\frac{1}{2-(-3)}\int (-x + 2x - 5)$

$= -\frac{11}{2}$

Examples for Finding specific Antiderivatives Using initial Conditions

1) $y(x+1)+y'=0$ $y(-2)=1$ (Calc book, pg. 429 #15)

2) $yy'-e^x=0$ $y(0)=4$ (Notes sec. 6.2 Examp. #4)

3) Carbon 14 half life of 5730 years. Amount after 10,000 years is 2 grams.

4) Write the antiderivative F for $f(x)=\sin x$ so that $F(0)=3$

Examples for finding accumulative change from a rate of change

(1) Water is pumped into an underground tank at a constant rate of 8 gallons per min. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gal per min for $0 \le t \le 120$ min. At time $t=0$ the tank contains 30 gal of water

a) How many gallons of water leak out of the tank from $t=0$ to $t=3$ min?

b) Write an expression for A(t), the total number of gal of water in the tank at time t?

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Answer Key for Finding Specific Antiderivatives using Initial Conditions

1) $y(x+1)+y'=0 \quad y(-2)=1$

$y(x+1)=-\frac{dy}{dx}\cdot y$

$(x+1)=\frac{-dy}{dx}\cdot y\cdot dx$

$\int dx(x+1)=\int -dy\cdot y$

$\frac{x^2}{2}+x+C=-\frac{y^2}{2} \quad (-2,1)$

$\frac{(-2)^2}{2}-2+C=\frac{-1}{2}$

$C=-\frac{1}{2}$

$2\left(\frac{x^2}{2}+x-\frac{1}{2}\right)=\left(-\frac{y^2}{2}\right)2$

$x^2+2x-1=-y^2$

$\boxed{y^2=-x^2-2x+1}$

2) $yy'-e^x=0 \quad y(0)=4$

$yy'=e^x$

$dx\cdot y\frac{dy}{dx}=e^x\cdot dx$

$\int y\,dy=\int e^x\,dx$

$y^2=2e^x+C \quad (0,4)$

$16=2(e^0)+C$

$C=14$

$\boxed{y^2=2e^x+14}$

3) Carbon 14 has a half life of 5730 years. after 10,000 years, 2 grams is left.

$y=Ce^{kt}$

$\frac{1}{2}C=C\,e^{(5730)k}$

$\ln\frac{1}{2}=5730k$

$k=\frac{-\ln 2}{5730}$

$k=-.00012097$

$2=Ce^{(-.00012097)(10{,}000)}$

$C=\frac{2}{e^{-1.2097}} \qquad C=6.705g$

$\boxed{y=Ce^{-.00012097t}}$

1) $f'(x) = -\sin x$ so that $f(0) = 3$

$f(x) = \cos x + C$

$3 = \cos(0) + C$

$C = 2$

$\boxed{f(x) = 2 + \cos x}$

Answer Key for finding the accumulative rate of change from a rate of change

1) (a) The rate of change is $\int_0^3 \sqrt{t+1}\, dt$

$\sqrt{\frac{t^2}{2} + t}\,\Big]_0^3 = \boxed{\frac{14}{3} \text{ gallons}}$

(b) $A(0) = 36$ gall

$A'(t) = \text{rate in} - \text{rate out} \Rightarrow 8 - \sqrt{t+1}$

$A(t) = A(0) + \int_0^t A'(x)\, dx$

$36 + \int_0^t (8 - \sqrt{x+1}\, dx)$

$\boxed{36 + 8t - \int_0^t \sqrt{x+1}\, dx}$